On The Kantor Product

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Abstract. We study the algebra of bilinear multiplications of an n-dimensional vector space. In particular, we study the Kantor product of some well-known (associative, Lie, alternative, Novikov and some other) multiplications.

1. Introduction

Kantor introduced the class of conservative algebras in [11]. This class includes some well-known algebras: associative, Jordan, Lie, Leibniz and Zinbiel [13]. In the theory of conservative algebras of great importance is the conservative algebra $U(n)$ [12]. In the theory of Lie algebras $U(n)$ plays a role analogous to the role of $\mathfrak{gl}_n$. The space of the algebra $U(n)$ is the space of all bilinear multiplications on the n-dimensional space V_n. To define the operation of multiplication $[\ ,\ ]$ in the algebra $U(n)$ we fix a vector $u \in V_n$ and for two multiplications $A, B \in U(n)$ and two elements $x, y \in V_n$ we set

$$x * y = [A, B](x, y) = A(u, B(x, y)) - B(A(u, x), y) - B(x, A(u, y)). \tag{1}$$

Some properties of the algebra $U(2)$ were studied in [13, 14]. We say that the product of two multiplications on n-dimensional vector space defined by (1), is the left Kantor product of these multiplications. In a similar way we can define the right Kantor product and obtain similar results. We assume that the Kantor product is the left Kantor product. The Kantor product of a multiplication $\cdot$ by itself is *the Kantor square* of $\cdot$ and it is denoted by $[\cdot,\cdot]$. It gives us a map K from any variety V of algebras to some class $K(V)$.

The Kantor square of a multiplication $\cdot$ can be rewritten (see [11]) as the product of the left multiplication L_u and the multiplication $\cdot$, as $[L_u, \cdot]$, where

$$[L_u, \cdot](x, y) = u \cdot (x \cdot y) - (u \cdot x) \cdot y - x \cdot (u \cdot y) = [\cdot,\cdot](x, y).$$

The multiplication $[L_u, \cdot]$ plays an important role in the definition of a (left) conservative algebras [2, 11]. We recall that an algebra A with a multiplication $\cdot$ is called a (left) conservative algebra if and only if there exist a new multiplication $*$ such that

$$[L_a, [L_b, \cdot]] = -[L_{a*b}, \cdot].$$

The main aim of this paper is to study the properties of the Kantor product of multiplications. One of the central questions studied in this paper is the following:

Question. What identities does the class of algebras $K(V)$ satisfy if we know the identities of V?

We give some particularly answer of this question for associative, (anti)commutative, Perm, Lie, Leibniz, Zinbiel, left-commutative, bicommutative, Novikov, alternative, quasi-associative and quasi-alternative algebras; we also describe the Kantor product of multiplications in associative dialgebras, duplicial, dual duplicial, $As^{(2)}$, Poisson, generalized Poisson and Novikov-Poisson algebras. Finally, we study the Kantor square in some special cases; in particular, the associative algebras with identities, nilpotent and right-nilpotent algebras, associative algebras isomorphic to its Kantor square; and discuss the coincidence of derivations and automorphisms of the algebra and its Kantor square. Here we can to formulate

Open problem. Is $K(V)$ a variety of algebras for some variety V?

2. The Kantor square

In this section we leave technical and trivial proofs of lemmas. We are using the standard notation:

$$(a, b, c)_* = (a * b) * c - a * (b * c), (a, b, c) = (ab)c - a(bc);$$

$$[a, b]_* = a * b - b * a, [a, b] = ab - ba;$$

$$\circlearrowleft_{a,b,c} [f(a,b,c)] = f(a,b,c) + f(b,c,a) + f(c,a,b).$$

2.1. **Associative algebras.** The variety of *associative algebras* is defined by the identity

$$(xy)z = x(yz).$$

Lemma 1. *Let $(A;\cdot)$ be an associative algebra. Then $(A;[\cdot,\cdot])$ is an associative algebra.*

2.2. **(Anti)commutative algebras.** The variety of *(anti)commutative algebras* is defined by the identity

$$xy = \epsilon yx,$$

where $\epsilon = 1$ in the commutative case and $\epsilon = -1$ in the anticommutative case.

Lemma 2. *Let $(A;\cdot)$ be an (anti)commutative algebra. Then $(A;[\cdot,\cdot])$ is an (anti)commutative algebra.*

2.3. **Perm algebras.** The variety of *Perm algebras* (see, for example, [3]) is defined by the identity

$$(xy)z = x(yz) = x(zy).$$

Lemma 3. *Let $(A;\cdot)$ be a Perm algebra. Then $(A;[\cdot,\cdot])$ is a Perm algebra.*

2.4. **Lie algebras.** The variety of *Lie algebras* is defined by the identities

$$xy = -yx, (xy)z + (yz)x + (zx)y = 0.$$

Lemma 4. *Let $(A;\cdot)$ be a Lie algebra. Then $[\cdot,\cdot] = 0$.*

2.5. **Leibniz algebras.** The variety of (left) *Leibniz algebras* (see, for example, [6]) is defined by the identity

$$x(yz) = (xy)z + y(xz).$$

Lemma 5. *Let $(A;\cdot)$ be a (left) Leibniz algebra. Then $[\cdot,\cdot] = 0$.*

2.6. **Left-commutative algebras.** The variety of *left-commutative algebras* (see, for example, [16]) includes commutative-associative, bicommutative, Novikov, Zinbiel algebras and some other. This variety is defined by the identity

$$x(yz) = y(xz).$$

Lemma 6. *Let $(A;\cdot)$ be a left-commutative algebra. Then $(A;[\cdot,\cdot])$ is a left-commutative algebra.*

2.7. **Bicommutative algebras.** The variety of *bicommutative algebras* (see, for example, [8]) is defined by the identities

$$x(yz) = y(xz), (xy)z = (xz)y.$$

Lemma 7. *Let $(A;\cdot)$ be a bicommutative algebra. Then $(A;[\cdot,\cdot])$ is an associative-commutative algebra.*

2.8. **Zinbiel algebras.** The variety of (left) *Zinbiel algebras* (see, for example, [7]) is defined by the identity

$$x(yz) = (xy)z + (yx)z.$$

Lemma 8. *Let $(A;\cdot)$ be a (left) Zinbiel algebra. Then $(A;[\cdot,\cdot])$ is a (left) Zinbiel algebra.*

2.9. **Novikov algebras.** The variety of (left) *Novikov algebras* (see, for example, [9]) is defined by the identities

$$x(yz) = y(xz), (x, y, z) = (x, z, y).$$

Lemma 9. *Let $(A; \cdot)$ be a (left) Novikov algebra. Then $(A; [\cdot, \cdot])$ is a (left) Novikov algebra.*

2.10. **Alternative algebras.** The variety of *alternative algebras* (see, for example, [15]) is defined by the identities

$$x^2y = x(xy), xy^2 = (xy)y. \tag{2}$$

It is also well known (see, for example, [22]) that an alternative algebra is flexible:

$$(xy)x = x(yx);$$

and satisfies the Moufang identities:

$$x(yzy) = ((xy)z)y, (yzy)x = y(z(yx)), (xy)(zx) = x(yz);$$

and the following identities hold:

$$(x, y, z) = -(y, x, z), (x, y, z) = -(x, z, y),$$
$$(x, xy, z) = (x, y, z)x, (x, yx, z) = x(x, y, z).$$

The main example of a non-associative alternative algebra is a Cayley — Dickson algebra $\mathbf{C}$ [22]. Let F be a field of characteristic $\neq 2$. It is an algebra $\mathbf{C}$ with the basis $e_0 = 1, e_1, \dots, e_7$ and the following multiplication table:

1	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_1	$\alpha \cdot 1$	e_3	αe_2	e_5	αe_4	$-e_7$	$-\alpha e_6$
e_2	$-e_3$	$\beta \cdot 1$	$-\beta e_1$	e_6	e_7	βe_4	βe_5
e_3	$-\alpha e_2$	βe_1	$-\alpha\beta \cdot 1$	e_7	αe_6	$-\beta e_5$	$-\alpha\beta e_4$
e_4	$-e_5$	$-e_6$	$-e_7$	$\gamma \cdot 1$	$-\gamma e_1$	$-\gamma e_2$	$-\gamma e_3$
e_5	$-\alpha e_4$	$-e_7$	$-\alpha e_6$	γe_1	$-\alpha\gamma \cdot 1$	γe_3	$-\alpha\gamma e_2$
e_6	e_7	$-\beta e_4$	βe_5	γe_2	$-\gamma e_3$	$-\beta\gamma \cdot 1$	$-\beta\gamma \cdot e_1$
e_7	αe_6	$-\beta e_5$	$\alpha\beta e_4$	γe_3	$-\alpha\gamma e_2$	$\beta\gamma e_1$	$\alpha\beta\gamma \cdot 1$

(3)

Theorem 10. *Let $(A; \cdot)$ be an alternative algebra. Then $(A; [\cdot, \cdot])$ is a flexible algebra. Furthermore,*
1) $(A; [\cdot, \cdot])$ is an alternative algebra if and only if A satisfies the identity

$$(x, u, (x, u, y)) = 0; \tag{4}$$

2) $(A; [\cdot, \cdot])$ is a noncommutative Jordan algebra if and only if A satisfies the identity

$$[L_u L_x L_u L_x, R_u R_x] = [L_{xuxu}, R_{ux}];$$

3) $(A; [\cdot, \cdot])$ is a Jordan algebra if $(A, \cdot)$ is a commutative alternative algebra;
4) $(\mathbf{C}; [\cdot, \cdot])$ is alternative for a Cayley — Dickson algebra $\mathbf{C}$, if and only if $u = u_0 \cdot 1$.

Proof. It is easy to see that

$$a * b = u(ab) - (ua)b - a(ub) = (au)b - 2a(ub).$$

Now, we can see

$$(x * y) * x - x * (y * x) = ((xu)y - 2x(uy)) * x - x * ((yu)x - 2y(ux)) =$$
$$(((xu)y)u)x - 2((x(uy))u)x - 2((xu)y)(ux) + 4(x(uy))(ux) -$$
$$(xu)((yu)x) + 2(xu)(y(ux)) + 2x(u((yu)x)) - 4x(u(y(ux))) =$$
$$x(uyu)x - 2((x, u, y)u)x - 2x(uyu)x - 2x(uyu)x + 2((xu)y, u, x) + 4x(uyu)x -$$
$$x(uyu)x - 2(xy)(y, u, x) + 2x(uyu)x + 2x(uyu)x + 2x(u(y, u, x)) - 4x(uyu)x =$$
$$2[((xu)y, u, x) - (x, u, x(uy)) - ((xy)(y, u, x) - x(u(y, u, x)))] =$$
$$2[((x, u, y), u, x) - ((x, u, y), u, x)] = 0.$$

It follows that $(A,*)$ is a flexible algebra.

1) It is easy to see that a flexible algebra is alternative if and only if it satisfies the first identity from (2). We have

$$(x*x)*y - x*(x*y) =$$

$$-(xuxu)y + 2(xux)(uy) - (xu)((xu)y) + 2x(u((xu)y)) + 2(xu)(x(uy)) - 4x(u(x(uy))) =$$

$$-2(xu)^2y - 2x(u(x(uy))) + 2x(u((xu)y)) + 2(xu)(x(uy)) = (x,u,(x,u,y)).$$

Now, the multiplication $[\cdot,\cdot]$ is alternative if and only if $(x,u,(x,u,y)) = 0$.

2) Note that an algebra B is a *non-commutative Jordan algebra* if and only if it is flexible and it satisfies the Jordan identity: $(x^2,y,x) = 0$.

Obviously,

$$((x*x)*y)*x - (x*x)*(y*x) =$$

$$(-(xuxu)y + 2(xux)(uy))*x + (xux)*((yu)x - 2y(ux)) =$$

$$-(((xuxu)y)u)x + 2((xuxu)y)(ux) + 2(((xux)(uy))u)x - 4((xux)(uy))(ux) +$$

$$(xuxu)((yu)x) - 2(xux)(u((yu)x)) - 2(xuxu)(y(ux)) + 4(xux)(u(y(ux))) =$$

$$2(((xuxu)y)(ux) - (xuxu)(y(ux)) + ((x(u(x(uy))))u)x - x(u(x(u((yu)x)))).$$

Now, $[\cdot,\cdot]$ is a noncommutative Jordan multiplication if and only if

$$[L_xL_uL_xL_u, R_uR_x] = [L_{xuxu}, R_{ux}].$$

3) It is easy to see that if A is a commutative alternative algebra then we have

$$((xuxu)y)(ux) - (xuxu)(y(ux)) + (((xux)(uy))u)x - (xux)(u((yu)x)) =$$

$$(xu)((xu)((xu)y))) - (xu)((xu)((xu)y))) + x(u(x(u(x(uy))))) - x(u(x(u(x(uy))))) = 0.$$

It follows that $[\cdot,\cdot]$ is non-commutative Jordan and from Theorem 2 we infer that $[\cdot,\cdot]$ is Jordan.

4) If $(\mathbf{C},*)$ is an alternative algebra for every u then A satisfies (4). Note that for the elements $e_{i_1}, e_{i_2}, e_{i_3}$, where $e_{i_k}e_{i_l} \neq \epsilon e_{i_m}$ (where ϵ is some element from the ground field), we have

$$(e_{i_1}, e_{i_2}, (e_{i_1}, e_{i_2}, e_{i_3})) = 2(e_{i_1})^2(e_{i_2})^2e_{i_3}.$$

Such triple (i_1,i_2,i_3) we call a g-triple. It is easy to see that if (i,j,k) is not a g-triple then the subalgebra generated by e_i, e_j, e_k is a two-generated subalgebra, and by the Artin theorem this subalgebra is associative, i. e., $(e_i,e_j,(e_i,e_j,e_k)) = 0$. Now, for the element $u = u_0\cdot 1 + u_1e_1 + \ldots + u_7e_7$ we have $(e_i,u,(e_i,u,e_j)) = 0$ if and only if $\sum_{\text{k, (i,j,k) is a g-triple}} (u_ke_k)^2 = 0$. It is equivalent to the following system

$$\begin{array}{cccccccl}
\alpha u_1^2 & & -\alpha\beta u_3^2 & +\gamma u_4^2 & & -\beta\gamma u_6^2 & & = 0,\\
\alpha u_1^2 & +\beta u_2^2 & & & -\alpha\beta u_5^2 & -\beta\gamma u_6^2 & & = 0,\\
\alpha u_1^2 & +\beta u_2^2 & & +\gamma u_4^2 & & & +\alpha\beta\gamma u_7^2 & = 0,\\
\alpha u_1^2 & & -\alpha\beta u_3^2 & & -\alpha\beta u_5^2 & & +\alpha\beta\gamma u_7^2 & = 0,\\
 & +\beta u_2^2 & -\alpha\beta u_3^2 & +\gamma u_4^2 & -\alpha\beta u_5^2 & & & = 0,\\
 & +\beta u_2^2 & -\alpha\beta u_3^2 & & & -\beta\gamma u_6^2 & +\alpha\beta\gamma u_7^2 & = 0,\\
 & & & +\gamma u_4^2 & -\alpha\beta u_5^2 & -\beta\gamma u_6^2 & +\alpha\beta\gamma u_7^2 & = 0.
\end{array}$$

Calculating, we obtain $u_1 = \sqrt{\beta\gamma}u_7, u_6 = \sqrt{-\alpha}u_7, u_2 = u_3 = u_4 = u_5 = 0$.

Now, from the relation (4) by simple calculations (for example, for $x = e_1+e_2, y = e_1$ and $x = e_2+e_6, y = e_6$) we can find that $u_7 = 0$ and $u = u_0\cdot 1$.

The theorem is proved.

2.11. **Quasi-associative algebras.** *Quasi-associative algebras* (see, for example, [4]) is defined by the identities

$$(x,y,z)+(y,z,x)+(z,x,y)=0,$$

$$(x,y,z)=\alpha[y,[x,z]],$$

where α is a fixed element in the ground field F. It is known [4] that an algebra $(A,\cdot)$ is quasi-associative if and only if there exist an associative algebra A with the new multiplication, such that for some $\lambda \in F$:

$$x\cdot y=\lambda xy+(1-\lambda)yx.$$

Lemma 11. *Let $(A;\cdot)$ be a quasi-associative algebra. Then $(A;[\cdot,\cdot])$ is a quasi-associative algebra.*

2.12. **Quasi-alternative algebras.** *Quasi-alternative algebras* (see, for example, [4]) is defined by the identities

$$(x,y,x)=0,$$

$$(x,x,y)=\alpha[x,[x,y]],$$

where α is a fixed element from the ground field F. It is known [4] that an algebra $(A,\cdot)$ is a quasi-alternative algebra if and only there exist an alternative algebra A with new multiplication, such that for some $\lambda \in F$:

$$x\cdot y=\lambda xy+(1-\lambda)yx.$$

Lemma 12. *Let $(A;\cdot)$ be a quasi-alternative algebra. Then $(A;[\cdot,\cdot])$ is a flexible algebra.*

2.13. **Associative dialgebras.** The variety of *associative dialgebras* (see, for example, [20]) is defined by the identities

$$(x\vdash y)\vdash z=(x\dashv y)\vdash z, x\dashv(y\vdash z)=x\dashv(y\dashv z),$$

$$(x\vdash y)\vdash z=x\vdash(y\vdash z),(x\dashv y)\dashv z=x\dashv(y\dashv z),(x\vdash y)\dashv z=x\vdash(y\dashv z).$$

Lemma 13. *Let $(A;\vdash,\dashv)$ be an associative dialgebra. Then $(A;[\vdash,\dashv])$ is an associative algebra.*

2.14. **Duplicial algebras.** The variety of *duplicial algebras* (see, for example, [18]) is defined by the identities

$$(x\prec y)\prec z=x\prec(y\prec z),$$

$$(x\succ y)\prec z=x\succ(y\prec z),$$

$$(x\succ y)\succ z=x\succ(y\succ z).$$

Lemma 14. *Let $(A;\prec,\succ)$ be a duplicial algebra. Then $(A;[\succ,\prec])$ is an associative algebra.*

2.15. **Dual duplicial algebras.** The variety of *dual duplicial algebras* (see, for example, [23]) is defined by the identities

$$(x\prec y)\prec z=x\prec(y\prec z),(x\succ y)\prec z=x\succ(y\prec z),(x\succ y)\succ z=x\succ(y\succ z),$$

$$x\prec(y\succ z)=(x\prec y)\succ z=0.$$

Lemma 15. *Let $(A;\prec,\succ)$ be a dual duplicial algebra. Then $[\succ,\prec]=0$ and $(A;[\prec,\succ])$ is a 2-nilpotent algebra.*

2.16. **$As^{(2)}$-algebras.** The variety of *$As^{(2)}$-algebras* (see, for example, [23]) is defined by the identities

$$(x\circ y)\cdot z=x\circ(y\cdot z),(x\cdot y)\circ z=x\cdot(y\circ z),$$

$$(x\circ y)\circ z=x\circ(y\circ z),(x\cdot y)\cdot z=x\cdot(y\cdot z).$$

Lemma 16. *Let $(A;\cdot,\circ)$ be a $As^{(2)}$-algebra. Then $(A;[\cdot,\circ])$ and $(A;[\circ,\cdot])$ are associative algebras.*

2.17. **Commutative tridendriform algebra.** The variety of *commutative tridendriform algebras* (see, for example, [17]) is defined by the identities

$$x \cdot y = y \cdot x, (x \cdot y) \cdot z = x \cdot (y \cdot z),$$
$$(x \prec y) \prec z = x \prec (y \prec z) + x \prec (z \prec y),$$
$$(x \cdot y) \prec z = x \cdot (y \prec z).$$

Lemma 17. *Let $(A; \cdot, \prec)$ be a commutative tridendriform algebra. Then $(A; [\prec, \cdot])$ is a commutative algebra and $(A; [\cdot, \prec])$ is a right Zinbiel algebra.*

2.18. **Poisson algebras.** The variety of *Poisson algebras* (see, for example, [19]) is defined by the identities

$$xy = yx, (xy)z = x(yz), \{xy, z\} = \{x, z\}y + x\{y, z\},$$
$$\{x, y\} = -\{y, x\}, \{\{x, y\}, z\} + \{\{y, z\}, x\} + \{\{z, x\}, y\} = 0.$$

Theorem 18. *Let $(A; \cdot, \{,\})$ be a Poisson algebra. Then $[\{,\}, \cdot] = 0$ and $(A; [\cdot, \{,\}])$ is a Lie algebra.*

Proof. In the first case, we have

$$a * b = \{u, ab\} - \{u, a\}b - a\{u, b\} = 0.$$

In the second case,

$$a * b = u\{a, b\} - \{ua, b\} - \{a, ub\} = -a\{u, b\} - b\{a, u\} - u\{a, b\} = -b * a.$$

and

$$\begin{gathered}
(a * b) * c + (b * c) * a + (c * a) * b = \\
-(a\{u, b\} + \{a, u\}b + \{a, b\}u) * c \\
-(b\{u, c\} + \{b, u\}c + \{b, c\}u) * a \\
-(c\{u, a\} + \{c, u\}a + \{c, a\}u) * b = \\
\\
a\{u, b\}\{u, c\} + \{a\{u, b\}, u\}c + \{a\{u, b\}, c\}u+ \\
\{a, u\}b\{u, c\} + \{\{a, u\}b, u\}c + \{\{a, u\}b, c\}u+ \\
\{a, b\}u\{u, c\} + \{\{a, b\}u, u\}c + \{\{a, b\}u, c\}u+ \\
b\{u, c\}\{u, a\} + \{b\{u, c\}, u\}a + \{b\{u, c\}, a\}u+ \\
\{b, u\}c\{u, a\} + \{\{b, u\}c, u\}a + \{\{b, u\}c, a\}u+ \\
\{b, c\}u\{u, a\} + \{\{b, c\}u, u\}a + \{\{b, c\}u, a\}u+ \\
c\{u, a\}\{u, b\} + \{c\{u, a\}, u\}b + \{c\{u, a\}, b\}u+ \\
\{c, u\}a\{u, b\} + \{\{c, u\}a, u\}b + \{\{c, u\}a, b\}u+ \\
\{c, a\}u\{u, b\} + \{\{c, a\}u, u\}b + \{\{c, a\}u, b\}u = \\
\\
a\{u, b\}\{u, c\} + \{u, b\}\{a, u\}c + ca\{\{u, b\}, u\} + au\{\{u, b\}, c\} + \{u, b\}\{a, c\}u+ \\
\{a, u\}b\{u, c\} + \{\{a, u\}, u\}bc + \{a, u\}\{b, u\}c + bu\{\{a, u\}, c\} + \{a, u\}\{b, c\}u+ \\
\{a, b\}u\{u, c\} + \{\{a, b\}, u\}uc + \{a, b\}\{u, c\}u + uu\{\{a, b\}, c\}+ \\
b\{u, c\}\{u, a\} + \{b, u\}\{u, c\}a + ba\{\{u, c\}, u\} + \{\{u, c\}, a\}bu + \{u, c\}\{b, a\}u+ \\
\{b, u\}c\{u, a\} + \{b, u\}\{c, u\}a + ca\{\{b, u\}, u\} + \{b, u\}\{c, a\}u + \{\{b, u\}, a\}cu+ \\
\{b, c\}u\{a, u\} + \{\{b, c\}, u\}ua + \{b, c\}\{u, a\}u + uu\{\{b, c\}, a\}+ \\
c\{u, a\}\{u, b\} + \{c, u\}\{u, a\}b + cb\{\{u, a\}, u\} + \{c, b\}\{u, a\}u + cu\{\{u, a\}, b\}+ \\
\{c, u\}a\{u, b\} + \{\{c, u\}, u\}ab + \{c, u\}\{a, u\}b + \{c, u\}\{a, b\}u + \{\{c, u\}, b\}au+ \\
\{c, a\}u\{u, b\} + \{\{c, a\}, u\}ub + \{c, a\}\{u, b\}u + uu\{\{c, a\}, b\} = \\
\\
(a\{u, b\}\{u, c\} + \{c, u\}a\{u, b\}) + (\{u, b\}\{a, u\}c + \{a, u\}\{b, u\}c)+ \\
(ca\{\{u, b\}, u\} + ca\{\{b, u\}, u\}) + (\{u, b\}\{a, c\}u + \{c, a\}\{u, b\}u)+
\end{gathered}$$

$$(\{a,u\}b\{u,c\}+b\{u,c\}\{u,a\})+(\{\{a,u\},u\}bc+cb\{\{u,a\},u\})+$$
$$(\{a,u\}\{b,c\}u+\{b,c\}u\{a,u\})+(\{a,b\}u\{u,c\}+\{c,u\}\{a,b\}u)+$$
$$(\{a,b\}\{u,c\}u+\{b,u\}\{u,c\}a)+(ba\{\{u,c\},u\}+\{\{c,u\},u\}ab)+$$
$$(\{b,u\}c\{u,a\}+\{c,u\}\{u,a\}b)+(\{b,u\}\{c,a\}u+\{c,a\}u\{u,b\})+$$
$$(\{b,c\}\{u,a\}u+\{c,b\}\{u,a\}u)+(\{c,u\}\{u,a\}b+\{c,u\}\{a,u\}b)+$$
$$[au\{\{u,b\},c\}+au\{\{b,c\},u\}+au\{\{c,u\},b\}]+$$
$$[bu\{\{a,u\},c\}+bu\{\{u,c\},a\}+bu\{\{c,a\},u\}]+$$
$$[cu\{\{a,b\},u\}+cu\{\{u,a\},b\}+cu\{\{b,u\},a\}]+$$
$$[uu\{\{a,b\},c\}+uu\{\{b,c\},a\}+uu\{\{c,a\},b\}]=0.$$

The theorem is proved.

2.19. **Generalized Poisson algebras.** The variety of unital *generalized Poisson algebras* (see, for example, [1]) is defined by the identities

$$xy=yx,(xy)z=x(yz),\{xy,z\}=\{x,z\}y+x\{y,z\}+D(z)xy,D(x)=\{1,x\},$$
$$\{x,y\}=-\{y,x\},\{\{x,y\},z\}+\{\{y,z\},x\}+\{\{z,x\},y\}=0.$$

Theorem 19. *Let $(A;\cdot,\{,\})$ be a generalized Poisson algebra. Then $(A;[\{,\},\cdot])$ is an associative-commutative algebra, and $(A;[\cdot,\{,\}])$ is a Lie algebra.*

Proof. In the first case, we have

$$a*b=\{u,ab\}-\{u,a\}b-a\{u,b\}=-D(u)ab=b*a$$

and

$$(a*b)*c=D(u)^2abc=a*(b*c).$$

In the second case,

$$a*b=u\{a,b\}-\{ua,b\}-\{a,ub\}=$$
$$-a\{u,b\}-b\{a,u\}-u\{a,b\}-D(b)ua+D(a)ub=-b*a.$$

Here we use the proof of Theorem 18. It is easy to see that

$$\begin{aligned}
\circlearrowleft_{a,b,c}[\quad &(a*b)*c]=\circlearrowleft_{a,b,c}[(\{b,u\}a+\{b,a\}u+\{u,a\}b-D(b)ua+D(a)ub)*c]=\\
\circlearrowleft_{a,b,c}[\quad &\{c,u\}\{b,u\}a+\{c,b\}\{b,u\}a+\{c,\{b,u\}\}au-\{b,u\}D(c)au+\{u,a\}\{b,u\}c+\\
&\{u,\{b,u\}\}ac-\{b,u\}D(u)ac-\{b,u\}D(c)au+\{b,u\}D(a)cu+D(\{b,u\})acu+\\
&\{c,u\}\{b,a\}u+\{c,u\}\{b,a\}u+\{c,\{b,a\}\}u^2-\{b,a\}D(c)u^2+\{u,\{b,a\}\}cu-\\
&\{b,a\}D(u)cu-\{b,a\}D(c)u^2+\{b,a\}D(u)cu+D(\{b,a\})cu^2+\{c,u\}\{u,a\}b+\\
&\{c,b\}\{u,a\}u+\{c,\{u,a\}\}bu-\{u,a\}D(c)bu+\{u,\{u,a\}\}bc+\{u,b\}\{u,a\}c-\\
&\{u,a\}D(u)bc-\{u,a\}D(c)bu+\{u,a\}D(b)cu+D(\{u,a\})bcu-\{c,u\}D(b)au-\\
&\{c,D(b)\}au^2-\{c,u\}D(b)ua-\{c,a\}D(b)u^2+2D(b)D(c)au^2-\{u,a\}D(b)cu-\\
&\{u,D(b)\}acu+2D(b)D(u)acu+D(b)D(c)au^2-D(D(b))acu^2-D(b)D(u)acu-\\
&D(a)D(b)acu^2+\{c,u\}D(a)bu+\{c,D(a)\}bu^2+\{c,u\}D(a)bu+\{c,b\}D(a)u^2-\\
&2D(a)D(c)bu^2+\{u,D(a)\}bcu+\{u,b\}D(a)cu-2D(a)D(u)bcu-D(a)D(c)bu^2+\\
&D(D(a))bcu^2+D(a)D(u)bcu+D(a)D(b)cu^2].
\end{aligned}$$

By the proof of Theorem 18, we can conclude that the sum of all elements without D is zero. Now, we have

$$\begin{aligned}
\circlearrowright_{a,b,c} [\quad & (a*b)*c] = \\
\circlearrowright_{a,b,c} [\quad & -\{b,u\}D(c)au - \{b,u\}D(u)ac - \{b,u\}D(c)au + \{b,u\}D(a)cu + D(\{b,u\})acu - \\
& \{b,a\}D(c)u^2 - \{b,a\}D(u)cu - \{b,a\}D(c)u^2 + \{b,a\}D(u)cu + D(\{b,a\})cu^2 - \\
& \{u,a\}D(c)bu - \{u,a\}D(u)bc - \{u,a\}D(c)bu + \{u,a\}D(b)cu + D(\{u,a\})bcu - \\
& \{c,u\}D(b)au - \{c,D(b)\}au^2 - \{c,u\}D(b)ua - \{c,a\}D(b)u^2 + 2D(b)D(c)au^2 - \\
& \{u,D(b)\}acu + 2D(b)D(u)acu + D(b)D(c)au^2 - D(D(b))acu^2 - D(b)D(u)acu - \\
& \{u,a\}D(b)cu - D(a)D(b)acu^2 + \{c,u\}D(a)bu + \{c,D(a)\}bu^2 + \{c,u\}D(a)bu + \\
& \{c,b\}D(a)u^2 - 2D(a)D(c)bu^2 + \{u,D(a)\}bcu + \{u,b\}D(a)cu - 2D(a)D(u)bcu - \\
& D(a)D(c)bu^2 + D(D(a))bcu^2 + D(a)D(u)bcu + D(a)D(b)cu^2].
\end{aligned}$$

It is easy to see that

$$\begin{aligned}
\circlearrowright_{a,b,c} \quad & [D(D(a))bcu^2 - D(D(b))acu^2] = 0, \\
\circlearrowright_{a,b,c} \quad & [D(\{b,u\})acu + D(\{u,a\})bcu] = 0, \\
\circlearrowright_{a,b,c} \quad & [\{u,D(a)\}bcu - \{u,D(b)\}acu] = 0, \\
\circlearrowright_{a,b,c} \quad & [D(\{b,a\})cu^2 - \{c,D(b)au^2 + \{c,D(a)\}bu^2] = 0, \\
\circlearrowright_{a,b,c} \quad & [-2\{b,a\}D(c)u^2 - \{c,a\}D(b)u^2 + \{c,b\}D(a)u^2] = 0, \\
\circlearrowright_{a,b,c} \quad & [\{b,u\}D(c)au + \{u,a\}D(c)bu + \{c,u\}D(b)au + \{u,c\}D(a)bu] = 0.
\end{aligned}$$

Obviously, $\circlearrowright_{a,b,c} [(a*b)*c] = 0$ and $[\cdot,\{,\}]$ is a Lie algebra. The theorem is proved.

2.20. **Novikov-Poisson algebras.** The variety of left *Novikov-Poisson* algebras is defined by the identities

$$xy = yx, (xy)z = x(yz),$$
$$x \circ (y \circ z) = y \circ (x \circ z), (x,y,z)_\circ = (x,z,y)_\circ,$$
$$x \circ (yz) = (x \circ y)z, (xy) \circ z - x(y \circ z) = (xz) \circ y - x(z \circ y).$$

Theorem 20. *Let $(A;\cdot,\circ)$ be a left Novikov-Poisson algebra. Then $(A;[\cdot,\circ])$ is a left Novikov algebra and $(A;[\circ,\cdot])$ is an associative-commutative algebra.*

Proof. Firstly, we have

$$a*b = u(a\circ b) - (ua)\circ b - a\circ(ub) = -(ua)\circ b.$$

Hence,

$$a*(b*c) = (ua)\circ((ub)\circ c) = (ub)\circ((ua)\circ c) = b*(a*c),$$

and

$$\begin{aligned}
(a,b,c)_* = (a*b)*c - a*(b*c) = (u((ua)\circ b))\circ c - (ua)\circ((ub)\circ c) = \\
((ua)\circ(ub))\circ c - (ua)\circ((ub)\circ c) = (ua,ub,c)_\circ = (ua,c,ub)_\circ = \\
= ((ua)\circ c)\circ(ub) - (ua)\circ(c\circ(ub)) = \\
u(ua,c,b)_\circ = u(ua,b,c)_\circ = (a*c)*b - a*(c*b) = (a,c,b)_*.
\end{aligned}$$

Secondly,

$$a*b = u\circ(ab) - (u\circ a)b - a(u\circ b) = -u\circ(ab) = b*a.$$

Therefore,

$$(a*b)*c = u\circ((u\circ(ab))c) = u\circ(u\circ(abc)) = u\circ(a(u\circ(bc))) = a*(b*c).$$

The theorem is proved.

Similarly, the variety of right Novikov-Poisson algebras may be defined (see, for example, [21]). It is easy to prove the following theorem:

Theorem 21. *Let $(A;\cdot,\circ)$ be a right Novikov-Poisson algebra. Then $(A;[\cdot,\circ])$ is a right Novikov algebra and $(A;[\circ,\cdot])$ is a commutative algebra.*

3. The Kantor square of algebras of special type.

Here we study some special cases of the Kantor square. For an algebra $A := (A;\cdot)$ its the Kantor square $(A;[\cdot,\cdot])$ we denote by $(A,*)$. We denote the Kantor square for a fixed element u by $(A,*_u)$. We consider the relations between the ideals in A and $(A,*)$, the relations between an associative algebra A with polynomial identity and its the Kantor square. Moreover, the relations between the nilpotency and right nilpotency in A and $(A,*)$ are investigated.

3.1. **Ideals in the Kantor product.**

Theorem 22. *Let I be an ideal of A. Then $(I,*)$ is an ideal of $(A,*)$, but the converse statement is not true in general.*

Proof. It is easy to see that if $i \in I$ and $a \in A$ then

$$i * a = u(ia) - (ui)a - i(ua) \in I \text{ and } a * i = u(ai) - (ua)i - a(ui) \in I.$$

It follows that I is an ideal of $(A,*)$.

Conversely, we can consider the trivial case, where for an algebra A has zero Kantor square (for example, Lie or Leibniz algebra) and every subspace of A is an ideal of $(A,*)$. For the non-trivial case (nonzero Kantor product), we can consider the following associative algebra: $A_1 \oplus A_2$ is the direct sum of the matrix algebras of order 2. Here, if e_i is the unit of A_i then the subspace generated by A_1 and e_2 is an ideal of $(A,*_{e_1})$, but is not an ideal of $A_1 \oplus A_2$. The theorem is proved.

3.2. **Associative algebras with polynomial identity.**

Given a polynomial f in n variables, we define $f_*(x_1,\dots,x_n)$ as the value of f in $(A,*)$, where $x_1,\dots,x_n$ are some elements in A.

Theorem 23. *Let $(A;\cdot)$ be an associative algebra that satisfies the polynomial identity $f(x_1,\dots,x_n)$. Then there exists an identity g such that A and $(A,*)$ satisfy g.*

Proof. It is easy to see that if A satisfies the identity $f(x_1,\dots,x_n)$ then A satisfies the identity $g(x_1,\dots,x_n,z) = f(x_1,\dots,x_n)z$. By Theorem 1, the multiplication in algebra $(A,*)$ is defined by $x*y = -xuy$. Now, we can calculate the element $g_*(x_1,\dots,x_n,z)$ in $(A,*)$. Obviously, it is $(-1)^n f(x_1u,\dots,x_nu)z$ which amounts to zero in A. It follows that $(A,*)$ satisfies the identity g. The theorem is proved.

One of the most popular identity in the associative algebras is the standard polynomial identity of degree n:

$$s_n(x_1,\dots,x_n) = \sum_{\sigma\in S_n} (-1)^\sigma x_{\sigma(1)}\dots x_{\sigma(n)}.$$

Theorem 24. *Let $(A;\cdot)$ be an associative algebra that satisfies s_n. Then $(A,*)$ satisfies s_{n+1}.*

Proof. It is easy to see that the standard polynomial of degree $n+1$ may be written as

$$s_{n+1}(x_1,\dots,x_{n+1}) = \sum_{i=1}^{n}\Big(\sum_{\substack{\sigma\in S_{n+1},\\ \sigma(n+1)=i}} (-1)^\sigma x_{\sigma(1)}\dots x_{\sigma(n)}x_{\sigma(n+1)=i}\Big) =$$

$$= \sum_{i=1}^{n}(\epsilon_i s_n(x_1,\dots,\hat{x}_i,\dots,x_{n+1})x_i), \epsilon_i = \pm 1.$$

Now, by the proof of Theorem 23, $(A,*)$ satisfies the standard polynomial identity of degree $n+1$. The theorem is proved.

3.3. **Nilpotent algebras.** For the nilpotent algebras, we can prove the following theorem.

Lemma 25. *Let $(A;\cdot)$ be a nilpotent algebra of nilpotency index n. Then $(A,*)$ is a nilpotent algebra of nilpotency index $\leq [n/2]+1$.*

Proof. Obviously, every product of the form $x_1 * x_2 * \ldots * x_t$ (with some order of brackets) is a sum of multiplications of the form $y_1 y_2 \cdots y_{2t-1}$ (with some order of brackets). Now, it is easy to see that $(A,*)$ is nilpotent and its index of nilpotency $\leq [n/2]+1$. The Lemma is proved.

3.4. **Right-nilpotent algebras.** An algebra A is called right-nilpotent (or left-nilpotent) of nilpotency index n if it satisfies the identity

$$(\ldots(x_1x_2)\ldots)x_n = 0 \quad (\text{ or } x_1(\ldots(x_{n-1}x_n)\ldots) = 0).$$

Curiously, an analogue of the Theorem 25 is not true for the right-nilpotent algebras.

Theorem 26. *There exists a right nilpotent algebra $(A;\cdot)$ such that $(A,*_u)$ is not right nilpotent, but $(A,*_u)$ is solvable.*

Proof. An algebra A is *right alternative* if the following identity holds in A:

$$(x,y,y) = 0.$$

It is interesting fact that in contrast to the algebras of many well-studied classes (Jordan, alternative, Lie and so on) a right nilpotent right alternative algebra need not be non-nilpotent. The corresponding example of a five-dimensional right nilpotent but not nilpotent algebra belongs to Dorofeev [5]. Its basis is $\{a,b,c,d,e\}$, and the multiplication is given by (zero products of basis vectors are omitted)

$$ab = -ba = ae = -ea = db = -bd = -c, ac = d, bc = e.$$

It is easy to see that

$$c *_a b = a(cb) - (ac)b - c(ab) = c.$$

Obviously, $c = (\ldots(c *_a b) *_a \ldots) *_a b \neq 0$, and $(A,*_a)$ is not right-nilpotent. It is easy to see that $A^2 \subseteq \langle c,d,e\rangle$ and $A *_a A \subseteq \langle c,d,e\rangle$, but $(A *_a A) *_a (A *_a A) = 0$, and $(A,*_a)$ is solvable.

The theorem is proved.

3.5. **Derivations.** Remember that a linear mapping D of an algebra A is called a derivation if it satisfies the relation $D(xy) = D(x)y + xD(y)$. By [12], an element a of an algebra A is called a *Jacobi* element if if satisfies the relation $a(xy) = (ax)y + x(ay)$. All Jacobi elements of A form a vector space, which is called the *Jacobi* space of A.

Lemma 27. *Let D be a derivation of both A and $(A,*)$. Then*

1) *If A has zero Jacobi space, then $D = 0$;*

2) *If D is invertible, then A is a left Leibniz algebra and $(A,*)$ is a zero algebra. In particularly, if A is a finite-dimensional algebra over a field of zero characteristic, then A is nilpotent.*

Proof. 1). By simple calculations, from $D(x*y) = D(x)*y + x*D(y)$, we have

$$D(u)(xy) = (D(u)x)y + x(D(u)y).$$

By the definition of the Jacobi space, we have $D = 0$.

2). By invertibility of mapping D and arbitrarity of element u, we infer that A is a left Leibniz algebra. By the Lemma 5, we imply that $(A,*)$ is zero algebra.

In [10] it was proved that a finite-dimensional Leibniz algebra over a field of characteristic zero which admitting an invertible derivation is nilpotent. The Lemma is proved.

3.6. **Automorphisms.** Remember that an invertible linear mapping ϕ of an algebra is called an automorphism if it satisfies the relation $\phi(xy) = \phi(x)\phi(y)$.

Lemma 28. *Let ϕ be an automorphism of both A and $(A, *)$. If A is an algebra with zero Jacobi space, then ϕ is the identity mapping.*

Proof. By simple calculations from $\phi(x * y) = \phi(x) * \phi(y)$, we have

$$(u - \phi(u))(xy) = ((u - \phi(u))x)y + x((u - \phi(u))y).$$

By the definition of the Jacobi space, we have that $\phi = id$. The Lemma is proved.

3.7. **Isomorphic Kantor squares.** Here we talk about the situation where algebra A and its Kantor square are isomorphic.

Theorem 29. *Let A be a finite-dimensional associative algebra. Then A is isomorphic to $(A, *)$, if and only if A is a skew field.*

Proof. Let f_u is an isomorphism between algebras A and $(A, *_u)$ and $f_u(xy) = f_u(x)*f_u(y) = -f_u(x)uf_u(y)$. If in A there are two elements u and v with zero product, for $x = f_u^{-1}(v)$ we have

$$f_u(f_x(ab)) = -f_u(f_x(a)xf_x(b)) = f_u(f_x(a))uvuf_u(f_x(b)) = 0.$$

Now, if there is a zero divisor, then the algebra A has zero multiplication. It is Well known that every finite-dimensional algebra without zero divisors is a skew field.

On the other side, for some fixed nonzero element u from a skew field A we define $f_u(a) = -au^{-1}$. It is an isomorphism between algebras A and $(A, *_u)$ for every nonzero element u. The theorem is proved.

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